AL/2024/11/E-I

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2024
General Certificate of Education (Adv. Level) Examination, 2024

උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

අමතර කියවීම් කාලය - මිනිත්තු 10 යි
மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number | | | | | | | |

Instructions:

* *This question paper consists of two parts.*
  Part A (Questions 1 - 10) and **Part B** (Questions 11 - 17).
* **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
* **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
* *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
* *You are permitted to remove **only*** **Part B** *of the question paper from the Examination Hall.*

For Examiners' Use only

(11) Higher Mathematics I		
Part	**Question No.**	**Marks**
A	1	
	2	
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	10	
B	11	
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	16	
	17	
	Total	

Total

In Numbers	
In Words	

Code Numbers

Marking Examiner	
Checked by: 1	
2	
Supervised by:	

[see page two

Part A

1. Factorize: $a^3(b-c) + b^3(c-a) + c^3(a-b)$.

2. Let a relation R be defined on the set of all integers $\mathbb{Z}$ by aRb if $a-b$ is divisible by both 3 and 5. Show that R is an equivalence relation on $\mathbb{Z}$ and write down the equivalence class of 0.

[see page three

0212

3. Let $f(x) = x^2 - 4x + 5$ for $x \geq 2$. Show that f is one-to-one and find the range of f. Also find $f^{-1}(x)$.

4. Let $a, b, c \in \mathbb{R}$. Show that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = (1 + abc)(a-b)(b-c)(c-a)$.

[see page four

5. Show that the tangents drawn to the parabola $y^2 = 4ax$ from the point $(-a, 0)$ are perpendicular.

6. Let $a, b \in \mathbb{R}$. Also, let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \begin{cases} \dfrac{a\left(\sqrt{x+1} - 2\right)}{x-3} & \text{if } x > 3, \\ e^{x-3} + bx & \text{if } x \leq 3. \end{cases}$

It is given that f continuous and the gradient of the tangent line drawn to the graph of f when $x = 0$ is $e^{-3} + 1$. Find the values of a and b.

[see page five

7. Let $f(x) = 2^x$ for $x \in \mathbb{R}$ and $g(x) = 1 + \ln(x+1)$ for $x > -1$.

Find the gradient of the tangent lines drawn to the graphs of f and g when $x = 0$.

Now, let h: $\mathbb{R} \to \mathbb{R}$ be the function defined by $h(x) = \begin{cases} 2^x & \text{if} \quad x \le 0, \\ 1 + \ln(x+1) & \text{if} \quad x > 0. \end{cases}$

Is h differentiable at $x = 0$? Justify your answer.

8. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x-y}{x+y} + \dfrac{y}{x}$.

[see page six

9. Let f be a real-valued function such that f'' is continuous on $[0, 1]$ and $f'(1) = f(1)$.

Show that $\int_0^1 \left\{ x f'(x) + x^2 f''(x) \right\} \mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$.

10. On the same diagram, sketch the curves whose polar equations are $r = 1 + \sin\theta$ and $r = 3\sin\theta$. Find the coordinates of the point of intersection of these two curves with $0 < \theta < \frac{\pi}{2}$.

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[see page seven

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கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2024
General Certificate of Education (Adv. Level) Examination, 2024

උසස් ගණිතය I
உயர் கணிதம் I
Higher Mathematics I

11 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let A, B and C be subsets of a universal set S. Stating clearly the laws of Algebra of sets that you use, show that

(i) $(B \cup C) \cap A' = (A' \cap B) \cup (C \cap A' \cap B')$,

(ii) $A \cap (B \cap C') = (A \cap B) \cap (A \cap C)'$.

(*b*) Among a group of 400 tourists, 103 preferred to visit only Anuradhapura, 32 preferred to visit only Kandy and 71 preferred to visit only Jaffna. 19 preferred to visit all three cities. 41 do not prefer to visit any of these three cities. 235 do not prefer to visit Kandy. 200 do not prefer to visit Jaffna.

Find the number of tourists who preferred to visit only Kandy and Jaffna.

12. (*a*) Let $a, b, c \in \mathbb{R}$ and $a, b, c > 0$.

Show that

(i) $\dfrac{a+b}{2} \geq \sqrt{ab}$,

(ii) $\dfrac{a^2+b^2}{c^2} + \dfrac{b^2+2c^2}{a^2} + \dfrac{2c^2+a^2}{b^2} \geq 2 + 4\sqrt{2}$.

(*b*) Find the value of λ such that, $\begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$ for some non-zero $u, v \in \mathbb{R}$.

Hence or otherwise, find the equation of the line through the origin which gets mapped onto itself, under the transformation defined by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

[see page eight

13. State and prove **De Moivre's Theorem** for a positive integral index.

Using De Moivre's theorem, show that $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ for $\theta \in \mathbb{R}$.

Hence, show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + \tan^4\theta}$ when $\theta \neq \dfrac{n\pi}{5}$ for $n \in \mathbb{Z}$.

Prove that $\tan\dfrac{\pi}{5}$ is a root of the equation $x^4 - 10x^2 + 5 = 0$.

Give the other roots of the equation $x^4 - 10x^2 + 5 = 0$ in the form $\tan\dfrac{k\pi}{5}$, stating the values of k.

Deduce that $\tan^2\dfrac{\pi}{5} + \tan^2\dfrac{2\pi}{5} = 10$.

14. (*a*) Let C_1 and C_2 be the curves given by $y = \left|x^2 - 1\right|$ and $y = x^2 - x|x|$ respectively. Sketch the region S, enclosed by the curves C_1 and C_2.

Show that the area of the region S is $\dfrac{2}{3\sqrt{3}}\left(1 + \sqrt{3}\right)$.

Also, find the volume of the solid generated by rotating S about the x-axis through 2π radians.

(*b*) Solve the differential equation $\left(x^2 + 1\right)\dfrac{dy}{dx} + xy = \sqrt{x^2 + 1}\,\sin x$ subject to the condition $y = 1$ when $x = 0$.

15. (*a*) Let $I_n = \displaystyle\int_0^a x^n\sqrt{a^2 - x^2}\,dx$ for $a > 0$, and $n = 0, 1, 2, 3, \ldots$.

Show that $I_n = \dfrac{(n-1)}{(n+2)}a^2 I_{n-2}$ for $n \geq 2$.

Hence, evaluate $\displaystyle\int_0^2 x^4\sqrt{4 - x^2}\,dx$.

(*b*) Find the Maclaurin series of $\sin x$ and $\ln(1 - x)$ in ascending powers of x upto and including the term in x^5.

Hence, obtain the Maclaurin series of $\sin 2x \ln(1 + 2x)$ in ascending powers of x upto and including the term in x^3.

Using this, find an approximate value for $\displaystyle\int_0^{\frac{1}{4}} \sin 2x \ln\left(\frac{1 + 2x}{1 - 2x}\right)dx$.

[*see page nine*

16. (i) Prove that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if and only if $c^2 = a^2m^2 + b^2$.

(ii) Show that the locus of the point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are at right angles, is given by $x^2 + y^2 = a^2 + b^2$.

(iii) A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at the points P and Q. Show that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

17. (*a*) Let $f(x) = \frac{1-\cos 2x}{3+\cos 2x}$ for $x \in \mathbb{R}$.

(i) Show that $0 \leq f(x) \leq 1$ for $x \in \mathbb{R}$.

(ii) Solve the equations $f(x) = 0$ and $f(x) = 1$, and sketch the graph of $y = f(x)$ for $0 \leq x \leq 2\pi$ indicating the turning points.

(*b*) The following table gives the values of $f(x) = e^{-x^2}$, correct to two decimal places, for values of x between 0 and 1:

x	0.00	0.25	0.50	0.75	1.00
$f(x) = e^{-x^2}$	1.00	0.94	0.78	0.57	0.37

Applying **Simpson's Rule**, find an approximate value for $\int_0^1 e^{-x^2}\,dx$.

An approximate value of $\int_0^1 \left(15 - e^{3-x^2}\right)dx$ is given to be 0.54. Find an approximate value for e^3.

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AL/2024/11/E-II

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இலங்கைப் பரீட்சைத் திணைக்களம்
Department of Examinations, Sri Lanka

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2024
General Certificate of Education (Adv. Level) Examination, 2024

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

පැය තුනයි
மூன்று மணித்தியாலம்
Three hours

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மேலதிக வாசிப்பு நேரம்	-	10 நிமிடங்கள்
Additional Reading Time	-	**10 minutes**

Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.

Index Number | | | | | | | |

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- ⁕ *This question paper consists of two parts.* **Part A** (Questions **1 - 10**) and **Part B** (Questions **11 - 17**).
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- ⁕ *You are permitted to remove* ***only*** **Part B** *of the question paper from the Examination Hall.*

- ⁕ *Statistical Tables will be provided.*
- ⁕ *g denotes the acceleration due to gravity.*

For Examiners' Use only

(11) Higher Mathematics II		
Part	Question No.	Marks
A	1	
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B	11	
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	17	
	Total	

Total

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In Words	

Code Numbers

Marking Examiner		
Checked by:	1	
	2	
Supervised by:		

[see page two

Part A

1. Let $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$, where $\alpha \in \mathbb{R}$. A couple comprises of the force $\mathbf{F}$ acting along the line $\mathbf{r} = a\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k})$ and the force $-\mathbf{F}$ acting along the line $\mathbf{r} = \mathbf{i} + b\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k})$, where $a, b \in \mathbb{R}$. If the vector moment of the couple is $\mathbf{i} - \mathbf{j} + \mathbf{k}$, find the values of α, a and b.

2. Relative to a fixed origin O, the position vectors of three points A, B and C are $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$ and **hence** find a unit vector perpendicular to the plane containing A, B and C.

[see page three

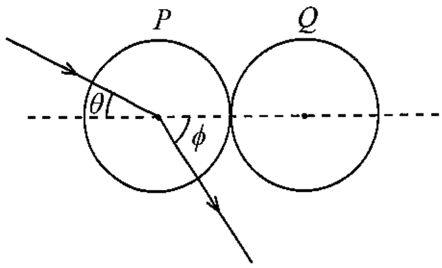
0212

3. A uniform solid right circular cylinder of radius a, height h and density ρ floats partially immersed in a homogeneous liquid of density 2ρ with its axis vertical. It is given that the immersed height of the cylinder is a. Show that $h = 2a$.
Also, find the force that should be applied vertically at the centre of the upper edge of the cylinder so that the immersed height of the cylinder is $\frac{3a}{2}$.

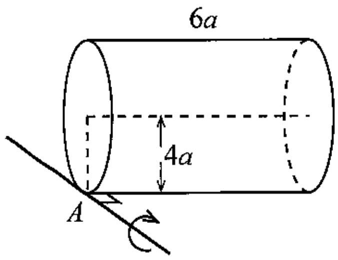
4. The velocity vector of a particle P at time t is given by $\mathbf{v}(t) = (\sin t + \cos t)\mathbf{i} + (t + t^2)\mathbf{j} + e^t\mathbf{k}$. It is also given that at $t = 0$, the position vector of P is equal to the acceleration vector of P. Find the position vector of P at time t.

[see page four

5. A smooth sphere P of mass m, moving at speed u, collides with an identical sphere Q which is initially at rest. Immediately before the collision, the velocity of P makes an angle θ with the line of centres. Immediately after the collision, the velocity of P makes an angle ϕ with the line of centres (see the figure). The coefficient of restitution between the spheres is $e\,(e<1)$. Show that $\tan\phi = \dfrac{2\tan\theta}{1-e}$.



6. A uniform solid right circular cylinder of mass M has radius $4a$ and height $6a$. It is given that the moment of inertia of the cylinder about an axis through the centre of mass perpendicular to the axis of the cylinder is $7Ma^2$. The cylinder can rotate about a fixed smooth horizontal axis through a point A on the circumference of one end of the cylinder. The axis of rotation is tangential to this circumference. The cylinder is held at rest with axis of the cylinder horizontal and at a height $4a$ above the point A (see the figure). Show that, if the cylinder is released, the angular speed at the instant when the axis of the cylinder is vertical is $\sqrt{\dfrac{7g}{16a}}$.



[see page five

7. A student randomly guesses answers to a multiple choice question paper. Each question has five choices, of which only one choice is correct. Assume that the student's choice for a question is independent of his choice for any other question.
Calculate the probability that
(i) answers to the first three questions are all correct.
(ii) at most two of the answers to the first three questions are correct.

8. The lifetime of a certain machine follows an exponential distribution. The probability that the machine does not fail during the first year of operation is 0.7. Find the probability that the machine will operate without failure for more than two years.
Given that the machine has not failed during its first year of operation, find the probability that it will continue to operate without failure for another year.

[see page six

9. The cumulative distribution function, $F(x)$ of a continuous random variable X is given by

$$F(x)=\begin{cases} kx(x+1) & , \text{ if } 0\leq x\leq 3, \\ 1 & , \text{ if } x>3, \\ 0 & , \text{ otherwise.} \end{cases}$$

Find the value of k, the median of X, and the mean of X.

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10. The probability mass function of a discrete random variable X is as follows:

x	0	1	2	3	4
$P(X=x)$	0.1	0.2	k	0.2	0.1

Find the value of k, the mean of X and the variance of X.
If the random variable Y, defined by $Y=aX-1$ has mean 0, find the value of a.

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[see page seven

අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2024
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2024
General Certificate of Education (Adv. Level) Examination, 2024

උසස් ගණිතය II
உயர் கணிதம் II
Higher Mathematics II

11 E II

Part B

* Answer **five** questions only.

11. (*a*) A force $\mathbf{F}$ produces a moment of $2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ about the origin O. The line of action of $\mathbf{F}$ lies on a plane perpendicular to the x axis and passes through the point $A \equiv (1, 1, 0)$.

(i) Determine $\mathbf{F}$.

(ii) It is given that the force $\mathbf{F}$ has zero moment about the point $B \equiv (p, 0, q)$. Find the values of p and q.

(*b*) Relative to a fixed origin O, points A and B of a rigid body have position vectors $b\mathbf{i}$ and $3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$ respectively. A force $\mathbf{F}$, acting on the body, passes through O and its line of action is parallel to AB. Forces $b\mathbf{i}-8\mathbf{j}+8\mathbf{k}$ and $p\mathbf{k}$, also acting on the body, pass through points with position vectors $b\mathbf{i}$ and $3\mathbf{i}+4\mathbf{j}$ respectively. Given that the system reduces to a couple, find $\mathbf{F}$ and p.

Find the couple in the form $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

12. A lamina in the shape of a rectangle $ABCD$ with $AB = a$ and $BC = b$ is immersed in a homogeneous liquid with its surface vertical and the edge AB on the free surface of the liquid. Show that the centre of pressure of the lamina is at a depth $\frac{2b}{3}$ from the free surface of the liquid.

A cubical tank has, on one of its vertical sides, a lid $PQRS$ in the shape of a square of side of length c which is smoothly hinged along the horizontal top edge PQ. The tank is filled with a homogeneous liquid of density ρ to the level of PQ. Find the magnitude of the minimum force that must be applied at the mid-point of RS perpendicular to the plane of $PQRS$ in order to keep the lid closed.

13. A particle P of mass m is projected vertically upwards under gravity with a initial velocity u. Assume that the resistance of air is gv^2 per unit mass, where v is the velocity of P. Obtain the equation of motion of P and show that P comes to rest at a height $\frac{1}{2g}\ln(1+u^2)$.

Show that the velocity ω with which the particle will return to the point of projection is given by $\omega = \frac{u}{\sqrt{u^2+1}}$.

[see page eight

14. (*a*)

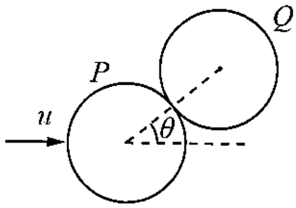


Figure (1)

A smooth sphere P moves with a speed u and makes a perfectly elastic collision with an identical sphere Q, which is at rest. Immediately before the collision, the velocity of P makes an angle θ with the line of centres (see figure (1)). Find the speeds of P and Q after the collisions and show that the directions of motion of P and Q after the collisions are perpendicular.

(*b*)

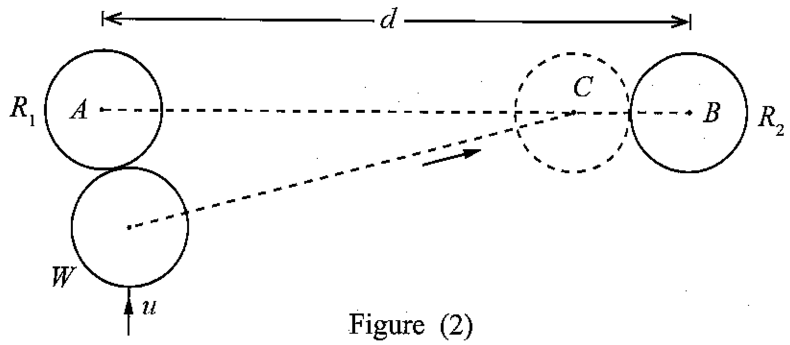


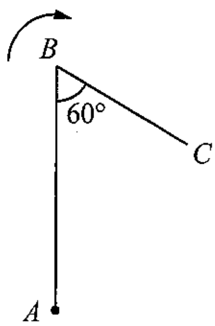
Figure (2)

In a billiard game, all the billiard balls are identical and of radius r. Assume that all the balls are smooth and the collisions are perfectly elastic.
Two red balls R_1 and R_2 are initially at rest with their centres at the points A and B respectively, where $AB = d(>3r)$. A white ball W moves with a speed u towards R_1 in a direction perpendicular to AB. After hitting R_1, W changes its direction and then collides with R_2. Suppose that at the moment of hitting R_2, the centre of W lies at a point C on AB (see figure (2)). Using the result in part(*a*), or otherwise,

(i) state the directions in which R_2 and W will move after the 2^{nd} collision.
and

(ii) find the speeds of R_2 and W after the 2^{nd} collision.

15. A frame consists of two uniform rods AB and BC, having lengths $4a$ and $2a$ respectively, which are rigidly joined together at B in such a way that $A\hat{B}C = 60°$, (see the figure). The mass per unit length of both rods is m. Show that the moment of inertia of the frame about an axis through A perpendicular to the plane of the frame is $48ma^3$.



The frame can rotate freely about a fixed horizontal axis through A and perpendicular to the frame. Initially AB is vertical, with B above A, and the frame is given an angular speed $\sqrt{\frac{3g}{8a}}$. The sense of the motion of the frame is such that, when AB is first horizontal, C is below AB. Show that angular speed of the frame at the instant when AB is horizontal is $\sqrt{\frac{\left(24+\sqrt{3}\right)g}{24a}}$.

[see page nine

16. A computer game consists of two levels, Level 1 and Level 2. A player starts at Level 1 and proceeds to Level 2, regardless of the outcome at Level 1. The probability of a win at Level1 is 0.6. The probability of a win at Level 2 is 0.4 or 0.2, depending on a win at Level1 or a loss at Level1 respectively.

(i) Find the probability that a player

(*a*) wins at both levels,

(*b*) wins only one of the two levels.

(ii) Given that a player has won at Level 2 of a game, find the probability that he has also won at Level1 of that game.

(iii) A player repeatedly plays this computer game until he wins at both levels of a game. Suppose that the outcomes of different games are independent. Find the expected number of games that a player has to play to win both levels of a game.

17. (*a*) A random variable X follows a Poisson distribution with the probability mass function

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots,$ where $\lambda (> 0)$ is a parameter.

Show that $E(X) = \lambda$ and $E(X^2) = \lambda^2 + \lambda$.

The number of misrecognized letters per page by a character recognition system, follows a Poisson distribution with a mean of 0.5. Find the variance of the number of misrecognized letters in a page.

Given that this system has misrecognized at most two letters in a particular page, find the probability that there are no misrecognized letters in that page.

(*b*) Of a certain production, the copper content in a product follows a normal distribution with a mean of 59.9 grams and a standard deviation of 2.5 grams. Find the probability that the copper content of a randomly selected product from this production is more than 61 grams.

Among the products with copper content less than 61 grams, find the percentage that will have copper content exceeding 60 grams.

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